Reflected Backward Stochastic Differential Equations Driven by Lévy Process

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Abstract: In this paper, we deal with a class of reflected backward stochastic differential equations associated to the subdifferential operator of a lower semi-continuous convex function driven by Teugels martingales associated with Lévy process. We obtain the existence and uniqueness of solutions to these equations by means of the penalization method. As its application, we give a probabilistic interpretation for the solutions of a class of partial differential-integral inclusions.

Keywords: Backward stochastic differential equation; partial differential-integral inclusion; Lévy process; Teugels martingale; penalization method.

Mathematics Subject Classification: 60H10; 60H30; 60H99.

1 Introduction

On the one hand, partial differential-integral inclusions (PDIIs in short) play an important role in characterizing many social, physical, biological and engineering problems, one can see Balasubramaniam and Ntouyas [3] and references therein. On the other hand, reflected backward stochastic differential equations (RBSDEs in short) associated to a multivalued maximal monotone

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operator defined by the subdifferential of a convex function has first been introduced by Gegout-Petit [6]. Further, Pardoux and Râcanu [18] gave the existence and uniqueness of the solution of RBSDEs, on a random (possibly infinite) time interval, involving a subdifferential operator was proved in order to to give the probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Following, Ouknine [16], N'Zi and Ouknine [13], Bahlali et al. [1, 2] discussed this type RBSDEs driven by Brownian motion or the combination of Brownian motion and Poisson random measure under the conditions of Lipschitz, locally Lipschitz or some monotone conditions on the coefficients. El Karoui et al. [5] have got another type RBSDEs different here, where one of the components of the solution is forced to stay above a given barrier, which provided a probabilistic formula for the viscosity solution of an obstacle problem for a parabolic PDE. Since then, there were many works on this topic. One can see Matoussi [12], Hamedène [8, 9], Lepeltier and Xu [11], Ren et al. [19, 20] and so on.

The main tool in the theory of BSDEs is the martingale representation theorem, which is well known for martingale which adapted to the filtration of the Brownian motion or that of Poisson point process (Pardoux and Peng [17], Tang and Li [22]). Recently, Nualart and Schoutens [14] gave a martingale representation theorem associated to Lévy process. Furthermore, they showed the existence and uniqueness of solutions to BSDEs driven by Teugels martingales associated with Lévy process with moments of all orders in [15]. The results were important from a pure mathematical point of view as well as in the world of finance. It could be used for the purpose of option pricing in a Lévy market and related partial differential equation which provided an analogue of the famous Black-Scholes partial differential equation.

Motivated by the above works, the purpose of the present paper is to consider RBSDEs associated to the subdifferential operator of a lower semi-continuous convex function driven by Teugels martingales associated with Lévy process which considered in Nualart and Schoutens [14, 15]. We obtain the existence and uniqueness of the solutions for RBSDEs. As its application, we give a probabilistic interpretation for a class of PDIIs.

The paper is organized as follows. In Section 2, we introduce some preliminaries and notations. Section 3 is devoted to the proof of the existence and

uniqueness of the solutions to RBSDEs driven by Lévy processes by means of the penalization methods. A probabilistic interpretation for a class of PDIIs by our RBSDEs is given in the last section.

2 Preliminaries and notations

Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $\{L_t : t \in [0, T]\}$ be a $\mathbb{R}$-valued Lévy process corresponding to a standard Lévy measure ν whose characteristic function has the following form:

$$E(e^{iuL_t}) = exp[iaut - \frac{1}{2}\sigma^2 u^2 t + t\int_{\mathbb{R}}(e^{iux} - 1 - iux1_{\{|x|<1\}})\nu(dx)],$$

where $a \in \mathbb{R}, \sigma \geq 0$. Furthermore, Lévy measure ν satisfying the following conditions:

(1) $\int_{\mathbb{R}}(1 \wedge y^2)\nu(dy) < \infty$;

(2) $\int_{]-\varepsilon,\varepsilon[^c} e^{\lambda|y|}\nu(dy) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$.

This shows that L_t has moments of all orders. We denote by $(H^{(i)})_{i\geq 1}$ the linear combination of so called Teugels Martingale $Y_t^{(i)}$ defined as follows associated with the Lévy process $\{L_t : t \in [0, T]\}$. More precisely

$$H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + \cdots + c_{i,1}Y_t^{(1)},$$

where $Y_t^{(i)} = L_t^{(i)} - E[L_t^{(i)}] = L_t^{(i)} - tE[L_1^{(i)}]$ for all $i \geq 1$ and $L_t^{(i)}$ are power-jump processes. That is, $L_t^{(i)} = L_t$ and $L_t^{(i)} = \sum_{0<s\leq t}(\triangle L_t)^i$ for $i \geq 2$. It was shown in Naulart and Schoutens [14] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \cdots$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$:

$$q_{i-1} = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

The martingale $(H^{(i)})_{i\geq 1}$ can be chosen to be pairwise strongly orthonormal martingale. Furthermore, $[H^{(i)}, H^{(j)}], i \neq j$, and $\{[H^{(i)}, H^{(j)}]_t - t\}_{t\geq 0}$ are uniformly integrable martingales with initial value 0, i.e. $< H^{(i)}, H(j) >_t = \delta_{ij}t$.

Remark 2.1. *If $\nu = 0$, we are in the classic Brownian case and all non-zero degree polynomials $q_i(x)$ will vanish, giving $H_t^{(i)} = 0, i = 2, 3, \cdots$. If μ only has mass at 1, we are in the Poisson case; here also $H_t^{(i)} = 0, i = 2, 3, \cdots$. Both cases are degenerate in this Lévy framework.*

Let $\mathcal{N}$ denote the totality of $P-$null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \sigma(L_s, 0 \leq s \leq t) \vee \mathcal{N}.$$

Let us introduce some spaces:

$\bullet\mathcal{H}^2 = \{\varphi : \mathcal{F}_t-$progressively measurable process, real-valued process, s.t. $E\int_0^T |\varphi_t|^2 dt < \infty\}$ and denote by $\mathcal{P}^2$ the subspace of $\mathcal{H}^2$ formed by the predictable processes;

$\bullet S^2 = \{\varphi : \mathcal{F}_t-$progressively measurable process, real-valued process, s.t. $E(\sup_{0\leq t\leq T} |\varphi(t)|^2) < \infty\}$;

$\bullet l^2 = \{(x_n)_{n\geq 0}$:be real valued sequences such that $\sum_{i=0}^{\infty} x_i^2 \leq \infty\}$. We shall denote by $\mathcal{H}^2(l^2)$ and $\mathcal{P}^2(l^2)$ the corresponding spaces of l^2-valued process equipped with the norm

$$||\varphi||^2 = \sum_{i=0}^{\infty} E\int_0^T |\varphi_t^{(i)}|^2 dt.$$

Now, we give the following assumptions:

(H1) The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, P)$;

(H2) The coefficient $f : [0, T] \times \Omega \times R \times l^2 \rightarrow R$ is $\mathcal{F}_t-$progressively measurable, such that $f(\cdot, 0, 0) \in \mathcal{H}^2$;

(H3) There exists a constant $C > 0$ such that for every $(\omega, t) \in \Omega \times [0, T], (y_1, z_1), (y_2, z_2) \in R \times l^2$

$|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + ||z_1 - z_2||^2)$;

(H4) Let $\Phi : R \rightarrow (-\infty, +\infty]$ be a proper lower semicontinuous convex function.

Define:

$Dom(\Phi) := \{x \in R : \Phi(x) < +\infty\}$,

$\partial\Phi(x) := \{x^* \in R :< x^*, x - u > +\Phi(u) \leq \Phi(x), \forall x \in R\}$,

$Dom(\partial\Phi)) := \{x \in R : \partial\Phi \neq \emptyset\}$,

$Gr(\partial\Phi)) := \{(x, x^*) \in R^2 : x \in Dom(\partial\Phi)), x^* \in \partial\Phi(x)\}$,

Now, we introduce a multivalued maximal monotone operator on R defined by the subdifferential of the above function Φ. The details appeared in Brezis [4].

For all $x \in R$, define

$\Phi_n(x) = \min_y(\frac{n}{2}|x-y|^2 + \Phi(y))$.

Let $J_n(x)$ be the unique solution of the differential inclusion $x \in J_n(x) + \frac{1}{n}\partial\Phi(J_n(x))$, which is called the resolvent of the monotone operator $A = \partial\Phi$. Then, we have

Proposition 2.1. *(1) $\Phi_n : R \to R$ is Lipschitz continuous;*

(2) The Yosida approximation of $\partial\Phi$ is defined by $A_n(x) := \nabla\Phi_n(x) = n(x - J_n(x)), x \in R$ which is monotonic and Lipschitz continuous and there exists $a \in interior(Dom(\Phi))$ and positive numbers R, C satisfies

$$< \nabla\Phi_n(z)^*, z - a > \geq R|A_n(z)| - C|z| - C, \text{ for all } z \in R \text{ and } n \in N; \quad (1)$$

(3) For all $x \in R$, $A_n(x) \in A(J_n(x))$.

Further, we assure that $\xi \in \overline{Dom(\Phi)}$ and $E\Phi(\xi) < \infty$.

This paper is mainly discuss the following reflected backward stochastic differential equations. In doing so, we first give its definition.

Definition 2.1. *By definition a solution associated with the above assumptions (ξ, f, Φ) is a triple $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ of progressively measurable processes such that*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, \ 0 \leq t \leq T, \quad (2)$$

and satisfying

(1) $(Y_t, Z_t)_{0\leq t\leq T} \in S^2 \times \mathcal{P}^2(l^2)$ and $\{Y_t, 0 \leq t \leq T\}$ is càdlàg (right continuous with left limits) and take values in $\overline{Dom(\Phi)}$;

(2) $\{K_t, 0 \leq t \leq T\}$ is absolutely continuous, $K_0 = 0$, and for all progressively measurable and right continuous process $\{(\alpha_t, \beta_t), 0 \leq t \leq T\} \in Gr(\partial\Phi)$, we have

$$\int_0^\cdot (Y_t - \alpha_t)(dK_t + \beta_t dt) \leq 0.$$

In order to obtain the existence and uniqueness of the solutions to (2), we consider the following penalization form of (2):

$$Y_t^n = \xi + \int_t^T [f(s, Y_s^n, Z_s^n) - A_n(Y_s^n)]ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i),n} dH_s^{(i)}, \ 0 \le t \le T, \tag{3}$$

where ξ, f satisfies the assumptions stated above and A_n is the Yosida approximation of the operator $A = \partial\Phi$. Since A_n is Lipschitz continuous, it is known from the result of [15], that Eq. (3) has a unique solution.

Set $K_t^n = -\int_0^t A_n(Y_s^n), \ 0 \le t \le T$. Our aim is to prove the sequence (Y^n, Z^n, K^n) convergence to a sequence (Y, Z, K) which is our desired solution to RBSDEs (2).

3 Existence and uniqueness of the solutions

The principal result of the paper is the following theorem.

Theorem 3.1. *Assume that assumptions on ξ, f and Φ hold, the RBSDEs (2) has a unique solution $(Y_t, Z_t, K_t)_{0\le t\le T}$. Moreover,*
$\lim_{n\to\infty} E \sup_{0\le t\le T} |Y_t^n - Y_t|^2 = 0,$
$\lim_{n\to\infty} E \int_0^T ||Z_t^n - Z_t||^2 dt = 0,$
$\lim_{n\to\infty} E \sup_{0\le t\le T} |K_t^n - K_t|^2 = 0,$
where (Y^n, Z^n) be the solution of Eq. (3).

In the sequel, $C > 0$ denotes a constant which can change from line to line.

The proof of the Theorem is divided into the following Lemmas.

Lemma 3.1. *Under the assumptions of Theorem 3.1, there exists a constant $C_1 > 0$ such that for all $n \ge 1$*

$$E(\sup_{0\le t\le T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + \int_0^T |A_n(Y_s^n)|ds) \le C_1.$$

Proof. By Itô formula, we have

$$\begin{aligned} |Y_t^n - a|^2 &= |\xi - a|^2 + 2\int_t^T (Y_s^n - a) f(s, Y_s^n, Z_s^n) ds \\ &\quad -2\int_t^T (Y_s^n - a) A_n(Y_s^n) ds - \int_t^T ||Z_s^n||^2 ds \\ &\quad -2\sum_{i=1}^{\infty} \int_t^T (Y_s^n - a) Z_s^{(i),n} dH_s^i. \end{aligned} \tag{4}$$

Taking expectation on the both sides and considering (1), we obtain

$$
\begin{array}{rl}
E|Y_t^n - a|^2 + E\int_t^T ||Z_s^n||^2 ds & \\
\le & E|\xi - a|^2 + 2E\int_t^T (Y_s^n - a) f(s, Y_s^n, Z_s^n) ds \\
& -2RE\int_t^T |A_n(Y_s^n)| ds + 2CE\int_t^T |Y_s^n| ds + 2C.
\end{array} \tag{5}
$$

Further, we get

$$
\begin{array}{rl}
E|Y_t^n - a|^2 + E\int_t^T ||Z_s^n||^2 ds + 2RE\int_t^T |A_n(Y_s^n)| ds & \\
\le & E|\xi - a|^2 + 2E\int_t^T (Y_s^n - a) f(s, Y_s^n, Z_s^n) ds \\
& +2CE\int_t^T |Y_s^n| ds + 2C \\
\le & E|\xi - a|^2 + CE\int_t^T |Y_s^n - a|^2 ds + \frac{1}{2}E\int_t^T ||Z_s^n||^2 ds + C,
\end{array} \tag{6}
$$

where we have used the elementary inequality $2ab \le \beta^2 a^2 + \frac{b^2}{\beta^2}$ for all $a, b \ge 0$. So, we have

$$
E|Y_t^n - a|^2 + \frac{1}{2}E\int_t^T ||Z_s^n||^2 ds \le C(1 + E\int_t^T |Y_s^n - a|^2 ds). \tag{7}
$$

Gronwall inequality shows

$$
E|Y_t^n - a|^2 \le C, \ \forall n. \tag{8}
$$

So that

$$
E|Y_t^n|^2 \le C, \ \forall n. \tag{9}
$$

From (6) and (7), it is easy to show

$$
E\int_0^T (||Z_s^n||^2 + |A_n(Y_s^n)|) ds \le C, \ \forall n. \tag{10}
$$

Bulkholder-Davis-Gundy inequality shows the desired result of Lemma 3.1. $\square$

Lemma 3.2. *Under the assumptions of Theorem 3.1, there exists a constant* $C_2 > 0$ *such that for all* $n \ge 1$

$$
E\int_0^T |A_n(Y_s^n)|^2 ds \le C_2.
$$

Proof. Without loss of generality, we assume $\Phi \geq 0, \Phi(0) = 0$. Let $\psi_n \triangleq \frac{\Phi_n}{n}$. For the convexity of ψ_n and Itô formula, we have

$$\begin{array}{rcl}\psi_n(Y_t^n) & \leq & \psi_n(\xi) + \int_t^T \nabla\psi_n(Y_s^n)[f(s, Y_s^n, Z_s^n) - A_n(Y_s^n)]ds \\ & & -\sum_{i=1}^{\infty}\int_t^T \nabla\psi_n(Y_s^n)Z_s^{(i),n}dH_s^i.\end{array} \quad (11)$$

Taking expectation on the both sides, we obtain

$$\begin{array}{rcl}E\psi_n(Y_t^n) & \leq & E\psi_n(\xi) + E\int_t^T \nabla\psi_n(Y_s^n)f(s, Y_s^n, Z_s^n)ds \\ & & -E\int_t^T \nabla\psi_n(Y_s^n)A_n(Y_s^n)ds \\ & = & E\psi_n(\xi) + E\int_t^T \nabla\psi_n(Y_s^n)f(s, Y_s^n, Z_s^n)ds \\ & & -\frac{1}{n}E\int_t^T |A_n(Y_s^n)|^2 ds.\end{array} \quad (12)$$

Using the elementary inequality $2ab \leq \beta a^2 + \frac{b^2}{\beta}$, we obtain

$$\begin{array}{rl}& E\psi_n(Y_t^n) + \frac{1}{n}E\int_t^T |A_n(Y_s^n)|^2 ds \\ \leq & E\psi_n(\xi) + \frac{1}{2n}E\int_t^T |A_n(Y_s^n)|^2 ds + \frac{1}{2n}E\int_t^T |f(s, Y_s^n, Z_s^n)|^2 ds.\end{array} \quad (13)$$

Further, we have

$$\begin{array}{rl}& E\psi_n(Y_t^n) + \frac{1}{n}E\int_t^T |A_n(Y_s^n)|^2 ds \\ \leq & E\psi_n(\xi) + \frac{1}{2n}E\int_t^T |A_n(Y_s^n)|^2 ds + \frac{1}{2n}E\int_t^T |f(s, Y_s^n, Z_s^n)|^2 ds \\ \leq & E\psi_n(\xi) + \frac{1}{2n}E\int_t^T |A_n(Y_s^n)|^2 ds + \frac{1}{2n}E\int_t^T |f(s, Y_s^n, Z_s^n)|^2 ds \\ \leq & E\psi_n(\xi) + \frac{1}{2n}E\int_t^T |A_n(Y_s^n)|^2 ds + \frac{C}{n}E\int_t^T |Y_s^n|^2 ds \\ & +\frac{C}{n}E\int_t^T ||Z_s^n||^2 ds + \frac{C}{n}E\int_t^T |f(s,0,0)|^2 ds.\end{array} \quad (14)$$

Lemma 3.1 shows

$$E\psi_n(Y_t^n) + \frac{1}{n}E\int_t^T |A_n(Y_s^n)|^2 ds \leq \frac{C}{n},$$

which implies the desired result. $\square$

Lemma 3.3. *Under the assumptions of Theorem 3.1, (Y^n, Z^n) be a Cauchy sequence in $S^2 \times \mathcal{P}^2(l^2)$.*

Proof. By Itô formula, we have

$$\begin{array}{rl}& |Y_t^n - Y_t^m|^2 + \int_t^T ||Z_s^n - Z_s^m||^2 ds \\ = & 2\int_t^T (Y_s^n - Y_s^m)[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)]ds \\ & -2\int_t^T (Y_s^n - Y_s^m)(A_n(Y_s^n) - A_m(Y_s^m))ds \\ & -2\sum_{i=1}^{\infty}\int_t^T (Y_s^n - Y_s^m)(Z_s^{(i),n} - Z_s^{((i),m})dH_s^i.\end{array} \quad (15)$$

For

$I = J_n + \frac{1}{n}A_n = J_m + \frac{1}{m}A_m,\ A_m \in \partial\Phi(J_m(Y_s^m)),\ A_n \in \partial\Phi(J_n(Y_s^n)),$

we can obtain

$-(Y_s^n - Y_s^m)(A_n(Y_s^n)ds - A_m(Y_s^m)) \le \frac{1}{4m}|A_n(Y_s^n)|^2 + \frac{1}{4n}|A_m(Y_s^m)|^2.$

So, we get

$$
\begin{aligned}
&E|Y_t^n - Y_t^m|^2 + E\int_t^T ||Z_s^n - Z_s^m||^2 ds \\
&\quad \le 2CE\int_t^T (|Y_s^n - Y_s^m|^2 + |Y_s^n - Y_s^m||Z_s^n - Z_s^m|)ds \\
&\quad\quad + E\int_t^T (\tfrac{1}{4m}|A_n(Y_s^n)|^2 + \tfrac{1}{4n}|A_m(Y_s^m)|^2)ds \\
&\quad \le 2CE\int_t^T [(1+\beta)|Y_s^n - Y_s^m|^2 + \tfrac{1}{\beta}|Z_s^n - Z_s^m|^2]ds \\
&\quad\quad + E\int_t^T (\tfrac{1}{4m}|A_n(Y_s^n)|^2 + \tfrac{1}{4n}|A_m(Y_s^m)|^2)ds.
\end{aligned} \tag{16}
$$

Choosing β such that $\frac{2C}{\beta} < \frac{1}{2}$, then

$\sup_{0\le t\le T} E|Y_t^n - Y_t^m|^2 + \frac{1}{2}E\int_0^T ||Z_s^n - Z_s^m||^2 ds \le C(\frac{1}{n} + \frac{1}{m}).$

Further, Bulkholder-Davis-Gundy inequality shows

$$E\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2 + \frac{1}{2}E\int_0^T ||Z_s^n - Z_s^m||^2 ds \le C(\frac{1}{n} + \frac{1}{m}), \tag{17}$$

which shows our desired result. □

In order to obtain the existence of solutions to Eq. (2), we give the following Lemma appeared in Saisho [21].

Lemma 3.4. *Let $\{K^{(n)}, n \in N\}$ be a family of continuous functions of finite variation on R^+. Assume that:*

(i) $\sup_n |K^{(n)}|_t \le C_t < \infty,\ 0 \le t < \infty;$

(ii) $\lim_{n\to\infty} K^{(n)} = K \in C([0, +\infty); R).$

Then K is of finite variation. Moreover, if $\{f^{(n)}, n \in N\}$ is a family of continuous functions such that $\lim_{n\to\infty} f^{(n)} = f \in C([0, +\infty); R)$, *then*

$\lim_{n\to\infty} \int_s^t < f_u^{(n)}, dK_u^{n)} >= \int_s^t < f_u, dK_u >,\ \forall 0 \le s \le t < \infty.$

Proof of Theorem 3.1 *Existence.* Lemma 3.3 shows that (Y^n, Z^n) is a Cauchy sequence in space $S^2 \times \mathcal{P}^2(l^2)$. We denote its limit as (Y, Z). At the same time, it is easy to verify that $(K^n)_{n\in N}$ converges uniformly in $L^2(\Omega)$ to the process $K_\cdot = \lim_{n\to\infty} \int_0^\cdot A_n(Y_s^n)ds$, that is $E\sup_{0\le t\le T} |K_t^n - K_t|^2 = 0$.

Denote $H^1(0, T; R)$ be the Sobolev space consisting of all absolutely continuous functions with derivative in $L^2(0, T)$. Lemma 3.2 shows

$$\sup_{n\in N} E||K^n||^2_{H^2(0,T;R)} < \infty,$$

which implies that the sequence K^n is bounded in $L^2(\Omega; H^1(0,T;R))$. So, there exists an absolutely continuous function $K \in L^2(\Omega; H^1(0,T;R))$ which is the weak limit of K^n. Further, $\frac{dK_t}{dt} = V_t$, where $-V_t \in \partial\Phi(Y_t)$.

In the following, we verify that (Y, Z, K) is the unique solution to Eq. (2). Taking a subsequence, if necessary, we can show

$\sup_{0\le t\le T} |Y^n_t - Y_t| \to 0,$

$\sup_{0\le t\le T} |K^n_t - K_t| \to 0.$

It follows that Y_t is càglàg and K_t is continuous. Further, let (α, β) be a càglàg process with values in $Gr(\partial\Phi)$, it holds

$$< J_n(Y^n_t) - \alpha_t, dK^n_t + \beta_t dt > \le 0.$$

Since J_n is a contraction and Y^n uniformly converges to Y on $[0,T]$. It follows that $J_n(Y^n_t)$ converges to $pr(Y)$ uniformly, where pr denotes the projection on $\overline{Dom(\Phi)}$. Lemma 3.4 shows

$$< pr(Y_t) - \alpha_t, dK_t + \beta_t dt > \le 0.$$

In order to complete the proof of the existence, we need to verify the following

$$P(Y_t \in \overline{Dom(\Phi)}, 0 \le t \le T) = 1.$$

For the right continuity of Y, it suffices to prove

$$P(Y_t \in \overline{Dom(\Phi)}) = 1,\ \ 0 \le t \le T.$$

If not so, there exists $0 < t < T$ and $B_0 \in \mathcal{F}$ such that $P(B_0) > 0$ and $Y_{t_0} \notin \overline{Dom(\Phi)}, \forall \omega \in B_0$. By the right continuity, there exists $\delta > 0,\ B_1 \in \mathcal{F}$ such that $P(B_1) > 0,\ Y_t \notin \overline{Dom(\Phi)}$ for $(\omega, t) \in B_1 \times [t_0, t_0 + \delta]$.

Using the fact $\sum_{n\in N} E \int_0^T |A_n(Y^n_s)| ds < \infty$ and Fatou Lemma, we get

$$\int_{B_1} \int_{t_0}^{t_0+\delta} \liminf_{n\to\infty} |A_n(Y^n_s)| ds dP < \infty,$$

which is impossible for $\liminf_{n\to\infty} |A_n(Y^n_s)| = \infty$ on the set $B_1 \times [t_0, t_0 + \delta]$.

Uniqueness. Let $(Y_t, Z_t, K_t)_{0\le t\le T}$ and $(Y'_t, Z'_t, K'_t)_{0\le t\le T}$ be two solutions for Eq. (2). Define

$$(\triangle Y_t, \triangle Z_t, \triangle K_t)_{0\le t\le T} = (Y_t - Y'_t, Z_t - Z'_t, K_t - K'_t)_{0\le t\le T}.$$

Itô formula shows

$$\begin{aligned} & E|\triangle Y_t|^2 + E\int_t^T ||\triangle Z_t||^2 ds \\ = \; & 2E\int_t^T \triangle Y_s[f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s)]ds + 2E\int_t^T \triangle Y_s d\triangle K_s. \end{aligned} \quad (18)$$

Since $\partial\Phi$ is monotone and $-\frac{dK_t}{dt} \in \partial\Phi(Y_t),\ -\frac{dK'_t}{dt} \in \partial\Phi(Y'_t)$, we obtain

$$E\int_t^T \triangle Y_s d\triangle K_s \leq 0.$$

Further,

$$E|\triangle Y_t|^2 + E\int_t^T ||\triangle Z_t||^2 ds \leq CE\int_t^T |\triangle Y_s|^2 ds + \frac{1}{2}E\int_t^T ||\triangle Z_s||^2 ds.$$

Gronwall inequality shows the uniqueness of the solutions to Eq. (2).

4 Applications

In this section, we study the link between RBSDEs driven Lévy processes and the solution of a class of PDIIs. Suppose that our Lévy process has the form of $L_t = at + X_t$ where X_t is a pure jump Lévy process with Lévy measure $\nu(dx)$.

In order to attain our main result, we give a Lemma appeared in [15].

Lemma 4.1. *Let $c : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$|c(s,y)| \leq a_s(y^2 \wedge |y|) \ a.s.,$$

where $\{a_s, s \in [0,T]\}$ is a non-negative predictable process such that $E\int_0^T a_s^2 ds < \infty$. Then, for each $0 \leq t \leq T$, we have

$$\begin{aligned} \sum_{t<s\leq T} c(s,\Delta L_s) \;=\; & \sum_{i=1}^{\infty}\int_t^T < c(s,\cdot),p_i >_{L^2(\nu)} dH_s^{(i)} \\ & + \int_t^T\int_{\mathbb{R}} c(s,y)\nu(dy)ds. \end{aligned}$$

Consider the following coupled RBSDEs:

$$Y_t = h(L_T) + \int_t^T f(s,L_s,Y_s,Z_s)ds + K_T - K_t - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)},\ 0 \leq t \leq T, \quad (19)$$

where $E|h(L_T)|^2 < \infty$.

Define

$$u^1(t,x,y) = u(t,x+y) - u(t,x) - \frac{\partial u}{\partial x}(t,x)y,$$

where u is the solution of the following PDIIs:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + a'\frac{\partial u}{\partial x}(t,x) + f(t,u(t,x),\{u^{(i)}(t,x)\}_{i=1}^{\infty}) \\ \qquad\qquad + \int_{\mathbb{R}} u^1(t,x,y)\nu(dy) \in \partial\Phi(u(t,x)), \\ u(T,x) = h(x) \in \overline{Dom(\Phi)}, \end{cases} \tag{20}$$

where $a' = a + \int_{\{|y|\geq 1\}} y\nu(dy)$ and

$$u^{(1)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t,x)(\int_{\mathbb{R}} y^2\nu(dy))^{\frac{1}{2}},$$

and for $i \geq 2$

$$u^{(i)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_i(y)\nu(dy).$$

Suppose that u is $\mathcal{C}^{1,2}$ function such that $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ is bounded by polynomial function of x, uniformly in t. Then we have the following

Theorem 4.1. *The unique adapted solution of (19) is given by*

$$\begin{aligned} &Y_t = u(t,L_t), \\ &K_t = \int_0^t V_s ds,\ -V_s \in \partial\Phi(u(t,L_t)), \\ &Z_t^{(i)} = \int_{\mathbb{R}} u^1(t,L_{t-},y)p_i(y)\nu(dy),\ i \geq 2, \\ &Z_t^{(1)} = \int_{\mathbb{R}} u^1(t,L_{t-},y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}(t,L_{t-})(\int_{\mathbb{R}} y^2\nu(dy))^{\frac{1}{2}}. \end{aligned}$$

Proof. Applying Itô formula to $u(s,L_s)$, we obtain

$$\begin{aligned} &u(T,L_T) - u(t,L_t) \\ &\quad = \int_t^T \frac{\partial u}{\partial s}(s,L_{s-})ds + \int_t^T \frac{\partial u}{\partial x}(s,L_{s-})dL_s \\ &\qquad + \sum_{t<s\leq T}[u(s,L_s) - u(s,L_{s-}) - \frac{\partial u}{\partial x}(s,L_{s-})\triangle L_s]. \end{aligned} \tag{21}$$

Lemma 4.1 applied to $u(s,L_{s-}+y) - u(s,L_{s-}) - \frac{\partial u}{\partial x}(s,L_{s-})y$ shows

$$\begin{aligned} &\sum_{t<s\leq T}[u(s,L_s) - u(s,L_{s-}) - \frac{\partial u}{\partial x}(s,L_{s-})\triangle L_s] \\ &= \sum_{i=1}^{\infty}\int_t^T(\int_{\mathbb{R}} u^1(s,L_{s-},y)p_i(y)\nu(dy))dH_s^{(i)} + \int_t^T\int_{\mathbb{R}} u^1(s,L_{s-},y)\nu(dy)ds. \end{aligned} \tag{22}$$

Note that

$$L_t = Y_t^{(1)} + tEL_1 = (\int_{\mathbb{R}} y^2\nu(dy))^{\frac{1}{2}}H_t^{(1)} + tEL_1, \tag{23}$$

where $EL_1 = a + \int_{\{|y|\geq 1\}} y\nu(dy)$.

Hence, substituting (22) and (23) into (21) yields

$$
\begin{aligned}
&h(L_T) - u(t, L_t)\\
&\quad = \int_t^T [\tfrac{\partial u}{\partial s}(s, L_{s-}) + a\tfrac{\partial u}{\partial x}(s, L_{s-}) + \int_{\{|y|\geq 1\}} y\nu(dy) + \int_{\mathbb{R}} u^1(s, L_{s-}, y)\nu(dy)]ds\\
&\qquad + \int_t^T [u^1(s, L_{s-}, y)p_1(y)\nu(dy) + \tfrac{\partial u}{\partial x}(s, L_{s-})(\int_{\mathbb{R}} y^2\nu(dy))^{\frac{1}{2}}]dH_s^{(1)}\\
&\qquad + \sum_{i=2}^{\infty} \int_t^T (\int_{\mathbb{R}} u^1(s, L_{s-}, y)p_i(y)\nu(dy))dH_s^{(i)}.
\end{aligned}
$$

From which we get the desired result of the Theorem. □

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